Question

Given that $\sum_{r=1}^{n} \frac{2r+1}{r^2(r+1)^2} = 1 - \frac{1}{(n+1)^2}$, use the substitution method to determine

(i) $\sum_{r=1}^{n} \frac{2r+5}{(r+2)^2(r+3)^2}$,

(ii) $\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2}$ in terms of n.

Solution

(i) Expressing the r-formula in terms of x,

$$\sum_{x=1}^{n} \frac{2x+1}{x^2(x+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Let $x = r+2$, when $x = 1$, $1 = r+2 \Rightarrow r = -1$
when $x = n$, $n = r+2 \Rightarrow r = n-2$

$$\sum_{r=-1}^{n-2} \frac{2(r+2)+1}{(r+2)^2(r+2+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Replacing all n with $n+2$,

$$\sum_{r=-1}^{n+2-2} \frac{2r+5}{(r+2)^2(r+3)^2} = 1 - \frac{1}{(n+2+1)^2}$$

Removing the terms with $r = -1$ and $r = 0$ from the summation,

$$\frac{2(-1)+5}{(-1+2)^2(-1+3)^2} + \frac{5}{2^2 3^2} + \sum_{r=1}^{n} \frac{2r+5}{(r+2)^2(r+3)^2} = 1 - \frac{1}{(n+3)^2}$$

$$\sum_{r=1}^{n} \frac{2r+5}{(r+2)^2(r+3)^2} = 1 - \frac{1}{(n+3)^2} - \frac{3}{4} - \frac{5}{36}$$

$$\sum_{r=1}^{n} \frac{2r+5}{(r+2)^2(r+3)^2} = \frac{1}{9} - \frac{1}{(n+3)^2}$$

(ii) Expressing the r-formula in terms of x,

$$\sum_{x=1}^{n} \frac{2x+1}{x^2(x+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Let $x = r-1$, when $x = 1$, $1 = r-1 \Rightarrow r = 2$
when $x = n$, $n = r-1 \Rightarrow r = n+1$

$$\sum_{r=2}^{n+1} \frac{2(r-1)+1}{(r-1)^2(r-1+1)^2} = 1 - \frac{1}{(n+1)^2}$$

Replacing all n with $n-1$,

$$\sum_{r=2}^{n-1+1} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{(n-1+1)^2}$$

$$\sum_{r=2}^{n} \frac{2r-1}{r^2(r-1)^2} = 1 - \frac{1}{n^2}$$