Maclaurin Series: Standard Expansion Examples

From $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^r}{r!} + ...$, this standard expansion can be applied to

$$e^{kx} = 1 + kx + \frac{(kx)^2}{2!} + \frac{(kx)^3}{3!} + \ldots$$

$$a^x = e^{x \ln a} = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \ldots$$

Other applications of the standard expansions include

$$\sin kx = kx - \frac{(kx)^3}{3!} + \frac{(kx)^5}{5!} - \ldots$$

$$\cos kx = 1 - \frac{(kx)^2}{2!} + \frac{(kx)^4}{4!} + \ldots$$

Example 1

$$\ln\left(\frac{1+x}{1-x}\right)$$

$$= \ln(1+x) - \ln(1-x)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\right)$$

$$\approx 2x + \frac{2x^3}{3}$$

Since the expansion of $\ln(1+x)$ is valid for $-1 < x \le 1$ and the expansion of $\ln(1-x)$ is valid for $-1 \le x < 1$, taking the intersection of the range of values of x , the expansion of $\ln\frac{1+x}{1-x}$ is valid for $-1 < x < 1$.

$$e^{\ln\left(\frac{1+x}{1-x}\right)}$$

$$\approx e^{2x + \frac{2x^3}{3}}$$

$$\approx 1 + \left(2x + \frac{2x^3}{3}\right) + \frac{\left(2x + \frac{2x^3}{3}\right)^2}{2!} + \frac{\left(2x + \frac{2x^3}{3}\right)^3}{3!} + \ldots$$

$$\approx 1 + 2x + \frac{2x^3}{3} + \frac{4x^2 + \frac{8x^4}{3}}{2} + \frac{8x^3}{6} + \ldots$$

$$\approx 1 + 2x + 2x^2 + 2x^3 + \ldots$$

Using Binomial Series to verify,

$$e^{\ln\left(\frac{1+x}{1-x}\right)} = \frac{1+x}{1-x} = -1 + \frac{2}{1-x}$$

$$= -1 + 2(1-x)^{-1}$$

$$\approx -1 + 2\left[1 + (-1)(-x) + \frac{(-1)(-2)}{2!}(-x)^2 + \frac{(-1)(-2)(-3)}{3!}(-x)^3 + \frac{(-1)(-2)(-3)(-4)}{4!}(-x)^4\right]$$

$$\approx -1 + 2(1 + x + x^2 + x^3)$$

$$\approx 1 + 2x + 2x^2 + 2x^3 + \ldots$$