Maclaurin Series

Standard Expansions

Functions	Expansion	Valid for
e^x	$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$	All values of x
$\sin x$	$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{(-1)^{r+1} x^{2r-1}}{(2r-1)!} + \ldots$	All values of x
$\cos x$	$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots + \frac{(-1)^{r+1} x^{2r-2}}{(2r-2)!} + \ldots$	All values of x
$\ln(1+x)$	$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots$	$-1 < x \le 1$
$(1+x)^n$	$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$	$-1 < x < 1$

Maclaurin Series

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

Derivation

1. e^x

Let $f(x) = e^x$ — When $x = 0$, $f(0) = e^0 = 1$

$f'(x) = e^x$ — $f'(0) = 1$

$f''(x) = e^x$ — $f''(0) = 1$

$f^3(x) = e^x$ — $f^3(0) = 1$

$f^4(x) = e^x$ — $f^4(0) = 1$

By substituting the values of $f(0)$, $f'(0)$, $f''(0)$, $f^3(0)$ and $f^4(0)$ back into Maclaurin Series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots$$

2. $\sin x$

Let $f(x) = \sin x$ — When $x = 0$, $f(0) = \sin 0 = 0$

$f'(x) = \cos x$ — $f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ — $f''(0) = -\sin x = 0$

$f^3(x) = -\cos x$ — $f^3(0) = -\cos x = -1$

$f^4(x) = \sin x$ — $f^4(0) = \sin 0 = 0$

By substituting the values back into Maclaurin Series,

$$\sin x = 0 + 1.x + 0x^2 + \frac{(-1)}{3!}x^3 + 0x^4 + \ldots = x - \frac{x^3}{3!} + \ldots$$

3. cos *x*

Let	$f(x) = \cos x$	When $x = 0$,	$f(0) = \cos 0 = 1$
	$f'(x) = -\sin x$		$f'(0) = -\sin x = 0$
	$f''(x) = -\cos x$		$f''(0) = -\cos x = -1$
	$f^3(x) = \sin x$		$f^3(0) = \sin 0 = 0$
	$f^4(x) = \cos x$		$f^4(0) = \cos 0 = 1$

By substituting the values back into Maclaurin Series,

$$\cos x = 1 + 0.x + \frac{(-1)}{2!}x^2 + 0.x^3 + \frac{1}{4!}x^4 + \ldots = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots$$

4. ln (1 + *x*)

Let	$f(x) = \ln(1+x)$	When $x = 0$,	$f(0) = \ln 1 = 0$
	$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1}$		$f'(0) = 1$
	$f''(x) = -(1+x)^{-2}$		$f''(0) = -1$
	$f^3(x) = 2(1+x)^{-3}$		$f^3(0) = 2$
	$f^4(x) = -6(1+x)^{-4}$		$f^4(0) = -6$

By substituting the values back into Maclaurin Series,

$$\ln(1+x) = 0 + 1.x + \frac{(-1)}{2}x^2 + \frac{2}{3!}x^3 + \frac{(-6)}{4!}x^4 + \ldots = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{r+1}x^r}{r} + \ldots$$

5. (1 + *x*)^n

Let

$$f(x) = (1+x)^n$$
$$f'(x) = n(1+x)^{n-1}$$
$$f''(x) = n(n-1)(1+x)^{n-2}$$
$$f^3(x) = n(n-1)(n-2)(1+x)^{n-3}$$
$$f^4(x) = n(n-1)(n-2)(n-3)(1+x)^{n-4}$$

When $x = 0$,

$$f(0) = (1)^n = 1$$
$$f'(x) = n$$
$$f''(x) = n(n-1)$$
$$f^3(x) = n(n-1)(n-2)$$
$$f^4(x) = n(n-1)(n-2)(n-3)$$

By substituting the values back into Maclaurin Series,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$