Maclaurin Series: Timed Practice Exam Questions' Solutions

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$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^3(0)}{3!}x^3 + \frac{f^4(0)}{4!}x^4 + \ldots + \frac{f^n(0)}{n!}x^n + \ldots$$

Question 1

$$y = \tan^{-1} x$$

$$\frac{dy}{dx} = \frac{1}{1+x^2} = \left(1+x^2\right)^{-1}$$

$$\frac{d^2y}{dx^2} = -\left(1+x^2\right)^{-2}.\,2x = \frac{-2x}{\left(1+x^2\right)^2}$$

$$\frac{d^3y}{dx^3} = \frac{-2\left(1+x^2\right)^2 - 2\left(1+x^2\right).2x.(-2x)}{\left(1+x^2\right)^4} = \frac{-2\left(1+x^2\right)^2 + 8x^2\left(1+x^2\right)}{\left(1+x^2\right)^4}$$

When $x = 0$,

$$y = \tan^{-1} 0 = 0$$

$$\frac{dy}{dx} = \frac{1}{1+0} = 1$$

$$\frac{d^2y}{dx^2} = \frac{-2(0)}{(1+0)^2} = 0$$

$$\frac{d^3y}{dx^3} = \frac{-2(1+0)^2 + 8(0)}{(1+0)^4} = -2$$

Using Maclaurin's theorem, the expression of $\tan^{-1} x$ in ascending powers of x is

$$\tan^{-1} x = 0 + 1.x + 0x^2 + \frac{-2}{3!}x^3 + \ldots$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \ldots$$

Question 2: N99/I/13

(i) $$y = \cos\left[\ln(1+x)\right]$$

Differentiating with respect to x,

$$\frac{dy}{dx} = -\sin\left[\ln(1+x)\right]\cdot\frac{1}{1+x}$$

$$(1+x)\frac{dy}{dx} = -\sin\left[\ln(1+x)\right]$$

Differentiating both sides with respect to x,

$$1.\frac{dy}{dx} + (1+x)\frac{d^2y}{dx^2} = -\cos\left[\ln(1+x)\right].\frac{1}{1+x}$$

$$(1+x)\frac{dy}{dx} + (1+x)^2\frac{d^2y}{dx^2} = -y$$

Differentiating both sides with respect to x ,

$$(1+x)^2\frac{d^3y}{dx^3}+2.(1+x).1.\frac{d^2y}{dx^2}+(1+x)\frac{d^2y}{dx^2}+1.\frac{dy}{dx}=-\frac{dy}{dx}$$

$$(1+x)^2\frac{d^3y}{dx^3}+3(1+x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

(ii) When $x=0$, $y=\cos[\ln(1+0)]=1$

$$(1+0)\frac{dy}{dx}=-\sin[0]\Rightarrow\frac{dy}{dx}=0$$

$$(1).0+1.\frac{d^2y}{dx^2}=-1\Rightarrow\frac{d^2y}{dx^2}=-1$$

$$\frac{d^3y}{dx^3}+3(-1)+2(0)=0\Rightarrow\frac{d^3y}{dx^3}=3$$

Thus, $$\cos[\ln(1+x)]=1+0.x+\frac{(-1)}{2}x^2+\frac{3}{3!}x^3+\dots$$

$$=1-\frac{1}{2}x^2+\frac{1}{2}x^3+\dots$$

(iii) If the standard series expansions for $\ln(1+x)$ and $\cos x$ are used,

$$\cos[\ln(1+x)]$$

$$=\cos\left[x-\frac{1}{2}x^2+\frac{1}{3}x^3+\dots\right]$$

$$=1-\frac{\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3+\dots\right)^2}{2}+\frac{\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3+\dots\right)^4}{4!}+\dots$$

$$=1-\frac{1}{2}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3+\dots\right)\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3+\dots\right)+\dots$$

$$=1-\frac{1}{2}\left(x^2-\frac{1}{2}x^3-\frac{1}{2}x^3\right)+\dots$$

$$=1-\frac{1}{2}x^2+\frac{1}{2}x^3+\dots$$

Question 3: DHS/2012/Promo/10

(i) $$y=\ln\left(\frac{1}{\sqrt{1+2x}}\right)=\ln(1+2x)^{-\frac{1}{2}}=-\frac{1}{2}\ln(1+2x)$$

Differentiating with respect to x,

$$\frac{dy}{dx}=-\frac{1}{2}\left(\frac{2}{1+2x}\right)=-\frac{1}{1+2x}$$

$$(1+2x)\frac{dy}{dx}=-1$$

Differentiating both sides with respect to x ,

$$(1+2x)\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$$

Differentiating both sides with respect to x ,

$$(1+2x)\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+2\frac{d^2y}{dx^2}=0$$
$$(1+2x)\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}=0$$

When $x=0$, $y=-\frac{1}{2}\ln 1=0$

$$\frac{dy}{dx}=-1$$
$$(1)\frac{d^2y}{dx^2}+2(-1)=0\Rightarrow\frac{d^2y}{dx^2}=2$$
$$(1)\frac{d^3y}{dx^3}+4(2)=0\Rightarrow\frac{d^3y}{dx^3}=-8$$

The Maclaurin's series for y up to and including the term in x^3 is

$$y=\ln\left(\frac{1}{\sqrt{1+2x}}\right)=0+(-1)x+\frac{2}{2}x^2+\frac{(-8)}{3!}x^3+\dots$$
$$=-x+x^2-\frac{4}{3}x^3+\dots$$

(ii) If the standard series expansions for $\ln(1+2x)$ is used,

$$y=\ln\left(\frac{1}{\sqrt{1+2x}}\right)$$
$$=-\frac{1}{2}\ln(1+2x)$$
$$=-\frac{1}{2}\left[2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}+\dots\right]$$
$$=-\frac{1}{2}\left[2x-2x^2+\frac{8}{3}x^3+\dots\right]$$
$$=-x+x^2-\frac{4}{3}x^3+\dots$$

The standard series expansion of $\ln(1+2x)$ is valid for $-1<2x\le 1$.

Thus, the range of values of x for $y=\ln\left(\frac{1}{\sqrt{1+2x}}\right)=-x+x^2-\frac{4}{3}x^3+\dots$ to be valid is $-\frac{1}{2}<x\le\frac{1}{2}$.

(iii) From **(i)** , when $x = -\frac{3}{8}$,

$$\ln\left(\frac{1}{\sqrt{1+2x}}\right) = -x + x^2 - \frac{4}{3}x^3 + \ldots$$

$$\ln\left(\frac{1}{\sqrt{1+2\left(-\frac{3}{8}\right)}}\right) = -\left(-\frac{3}{8}\right) + \left(-\frac{3}{8}\right)^2 - \frac{4}{3}\left(-\frac{3}{8}\right)^3 + \ldots$$

$$\ln\left(\frac{1}{\sqrt{\frac{1}{4}}}\right) \approx \frac{3}{8} + \frac{9}{64} + \frac{9}{128}$$

$$\ln 2 \approx \frac{75}{128} \text{ or } 0.586$$

(iv) From **(i)** , when $x = \frac{1}{2}$,

$$\ln\left(\frac{1}{\sqrt{1+2x}}\right) = -x + x^2 - \frac{4}{3}x^3 + \ldots$$

$$\ln\left(\frac{1}{\sqrt{1+2\left(\frac{1}{2}\right)}}\right) = -\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - \frac{4}{3}\left(\frac{1}{2}\right)^3 + \ldots$$

$$\ln\left(\frac{1}{\sqrt{2}}\right) \approx -\frac{1}{2} + \frac{1}{4} - \frac{1}{6}$$

$$-\frac{1}{2}\ln 2 \approx -\frac{5}{12}$$

$$\ln 2 \approx \frac{5}{6} \text{ or } 0.834$$

Though it is possible to approximate ln2 by substituting $x = \frac{1}{2}$ in the series expansion in **(i)**, $x = -\frac{3}{8}$ will give a better approximation as the magnitude of x is smaller. The smaller the value, the better the approximation of ln2 from a series of ascending powers of x .